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On separable states for composite systems of distinguishable fermions

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Abstract

We study separable (i.e., classically correlated) states for composite systems of spinless fermions that are distinguishable. For a proper formulation of entanglement formation for such systems, the state decompositions for mixed states should respect the univalence superselection rule. Fermion hopping always induces non-separability, while states with bosonic hopping correlation may or may not be separable. Under the Jordan–Klein–Wigner transformation from a given bipartite fermion system into a tensor product one, any separable state for the former is also separable for the latter. There are, however, $U(1)$-gauge invariant states that are non-separable for the former but separable for the latter.

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1. Introduction

We discuss the characterization of classically (i.e., separable) and non-classically correlated states for lattice fermion systems. We assume that fermion particles on different sites are distinguishable and compose a non-independent system. For such non-tensor-product quantum systems as well, we may consider state correlations among subsystems. See [1–3] that discuss the effect of fermion hopping terms to entanglement degrees of ground states for some multi-particle fermion lattice models (extended Hubbard model). We do not discuss the case of indistinguishable many fermions that are represented as anti-symmetric wavefunctions, see e.g. [4] and references therein.

Let $\mathbb{N}$ be a lattice of integers ordered by inclusion. The canonical anticommutation relations (CARs) are

$$\begin{aligned} \{a_i^\dagger, a_j\} &= \delta_{i,j}\mathbf{1}, \\ \{a_i^\dagger, a_j^\dagger\} &= \{a_i, a_j\} = 0, \qquad i, j \in \mathbb{N}, \end{aligned} \tag{1}$$

where $a_i^\dagger$ and a_i are creation and annihilation spinless fermion operators on the site i, and $\{A, B\} = AB + BA$. For each subset I of $\mathbb{N}$, the subsystem $\mathcal{A}(I)$ are generated by all $a_i^\dagger$ and a_i in I.

Let I and J be disjoint subsets of $\mathbb{N}$. We are interested in state correlations between the pair of subsystems $\mathcal{A}(I)$ and $\mathcal{A}(J)$. It is sometimes useful to convert quantum spin models to their corresponding lattice fermion models vice versa by Jordan–Klein–Wigner transformations. (But there are subtle points on this one-to-one correspondence when the degree of freedom is infinite and it is sometime impossible to translate the argument for the CAR (tensor-product) systems into that for the tensor-product (CAR) systems [5, 6].) We are going to compare the CAR and the tensor-product systems in terms of state correlations. Though we have no practical suggestion; we only refer to [7–9] for the topic of quantum computation using fermions, we hope that our mathematical study would be a basis for further investigation of quantum correlation for models coming from statistical mechanics, see e.g. [10–12] and references therein.

We give notation. The even–odd grading transformation is given by

$$\Theta\left(a_i^\dagger\right) = -a_i^\dagger, \qquad \Theta(a_i) = -a_i. \tag{2}$$

The even and odd parts of $\mathcal{A}(I)$ are

$$\mathcal{A}(I)_\pm := \{A \in \mathcal{A}(I) \mid \Theta(A) = \pm A\}.$$

We introduce $U(1)$-gauge transformation

$$\gamma_\theta\left(a_i^\dagger\right) = \mathrm{e}^{\mathrm{i}\theta} a_i^\dagger, \qquad \gamma_\theta(a_i) = \mathrm{e}^{-\mathrm{i}\theta} a_i \tag{3}$$

for $\theta \in \mathbb{C}^1$. A state that is invariant under Θ is called even and a state that is invariant under γ_θ for any $\theta \in \mathbb{C}^1$ is called $U(1)$-gauge invariant.

If the cardinality $|I|$ is finite, then $\mathcal{A}(I)$ is isomorphic to the $2^{|I|} \times 2^{|I|}$ full matrix algebra. Let

$$v_I := \prod_{i\in I} v_i, \qquad v_i := a_i^\dagger a_i - a_i a_i^\dagger. \tag{4}$$

This v_I gives an even self-adjoint unitary operator implementing Θ,

$$\mathrm{Ad}(v_I)(A) = \Theta(A), \qquad A \in \mathcal{A}(I). \tag{5}$$

The notion of separable states is unchanged for CAR systems: if a state is written as a convex sum of product states, then it is called a separable state. However, we note that due to the CAR structure (algebraic non-independence) there are limitations on marginal states that can be prepared on disjoint regions [13] and hence on product states.

According to the univalence superselection rule [14], any realizable state is Θ-invariant and thus noneven states are unphysical. But any even state has noneven-state decompositions (i.e., state decompositions in which there are noneven component states) unless it is pure. For a natural formulation of entanglement formation for even states of CAR systems, the state decompositions should be taken from the even-state space only, not from the whole state space. The entanglement formation under the univalence superselection rule is zero if and only if the given even state is separable (proposition 4).

We elucidate some characteristic properties on state correlations for fermion systems. Any fermionic particle hopping between disjoint subsystems always induces non-separability (proposition 1), while for tensor-product systems, states with particle hopping correlation may or not may be separable. We show that any separable state for the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$ is also separable for the tensor-product pair $(\mathcal{A}(I), \mathcal{A}(I)')$, where $\mathcal{A}(I)'$ denotes the commutant of $\mathcal{A}(I)$ in $\mathcal{A}(I \cup J)$ (proposition 3). It was already noted in [15] that the set of all separable states for the CAR pair is strictly smaller than that for the tensor-product pair. We reproduce this result by a more general argument. Also we show that this strict inclusion is realized in the $U(1)$-gauge invariant state space in section 4.

In section 5, we consider the general case including noneven states and provide a criterion of separability (proposition 6).

2. Separability condition for bipartite fermion systems

We give a definition of separability for fermion systems. Let I and J be mutually disjoint subsets of $\mathbb{N}$, and ω be a (not necessarily even) state on $\mathcal{A}(I \cup J)$. We denote the restriction of ω to $\mathcal{A}(I)$ ($\mathcal{A}(J)$) by ω_1 (ω_2). Conversely, we are given a pair of states ω_1 on $\mathcal{A}(I)$ and ω_2 on $\mathcal{A}(J)$. If there exists a state ω on the total system $\mathcal{A}(I \cup J)$ such that its restriction to $\mathcal{A}(I)$ is equal to ω_1 and that to $\mathcal{A}(J)$ is ω_2, then ω is called a state extension of ω_1 and ω_2. If

$$\omega(A_1A_2) = \omega_1(A_1)\omega_2(A_2) \tag{6}$$

for all $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$, then such ω is unique and called the product-state extension of ω_1 and ω_2 denoted as $\omega_1 \circ \omega_2$. The product property in the converse order, namely

$$\omega(A_2A_1) = \omega_2(A_2)\omega_1(A_1), \tag{7}$$

is a consequence of (6) combined with CARs and proposition 1.

We say that a state ω of $\mathcal{A}(I \cup J)$ satisfies the separability for the pair of subsystems $\mathcal{A}(I)$ and $\mathcal{A}(J)$, or ω is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$, if there exist a set of states $\{\omega_{1,i}\}$ on $\mathcal{A}(I)$, also that $\{\omega_{2,i}\}$ on $\mathcal{A}(J)$, and some positive numbers $\{\lambda_i\}$ such that $\sum_i \lambda_i = 1$, satisfying that

$$\omega(A_1A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1A_2) \tag{8}$$

for any $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$. This formula requires the existence of the product state $\omega_{1,i} \circ \omega_{2,i}$ for each pair of $\omega_{1,i}$ and $\omega_{2,i}$. For tensor-product systems, the existence of product-state extension for any given states on disjoint subsystems is automatic, while for fermion systems it is not always the case [13].

Proposition 1. *Let I and J be a pair of disjoint subsets and ω be a state on $\mathcal{A}(I \cup J)$. If ω is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then for any $A_{1-} \in \mathcal{A}(I)_-$ and $A_{2-} \in \mathcal{A}(J)_-$,*

$$\omega(A_{1-}A_{2-}) = 0. \tag{9}$$

If ω is a product state, then at least one of its restrictions to $\mathcal{A}(I)$ and $\mathcal{A}(J)$ is even.

Proof. First, we show the second statement. Let ω be a product state with its marginal states ω_1 on $\mathcal{A}(I)$ and ω_2 on $\mathcal{A}(J)$. Now suppose that both ω_1 and ω_2 are noneven. Hence, there are odd elements $A_{1-} \in \mathcal{A}(I)_-$ and $A_{2-} \in \mathcal{A}(J)_-$ such that $\omega_1(A_{1-}) \neq 0$ and $\omega_2(A_{2-}) \neq 0$. We are going to derive the contradiction. By the assumed product property,

$$\omega_1 \circ \omega_2(A_{1-}A_{2-}) = \omega_1(A_{1-})\omega_2(A_{2-}) \neq 0. \tag{10}$$

Both $A_{1-} + A_{1-}^{\dagger}$ and $\mathrm{i}\big(A_{1-} - A_{1-}^{\dagger}\big)$ are self-adjoint elements in $\mathcal{A}(I)_-$. Since A_{1-} can be written as their linear combination, the expectation value of at least one of them for ω_1 must be non-zero. Thus, we can take $A_{1-} = A_{1-}^{\dagger} \in \mathcal{A}(I)_-$ such that $\omega_1(A_{1-}) \neq 0$ and similarly $A_{2-} = A_{2-}^{\dagger} \in \mathcal{A}(J)_-$ such that $\omega_2(A_{2-}) \neq 0$.

Now both $\omega_1(A_{1-})$ and $\omega_2(A_{2-})$ are non-zero real, hence $\omega_1(A_{1-})\omega_2(A_{2-})$ is non-zero real. On the other hand, $A_{1-}A_{2-}$ is skew self-adjoint as

$$(A_{1-}A_{2-})^{\dagger} = A_{2-}^{\dagger}A_{1-}^{\dagger} = A_{2-}A_{1-} = -A_{1-}A_{2-}.$$

Thus, $\omega_1 \circ \omega_2(A_{1-}A_{2-})$ must be purely imaginary, which is a contradiction.

We assume that ω is a separable state. By definition, ω has a decomposition into the affine sum of product states:

$$\omega = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}.$$

Suppose that there exist $A_{1-} \in \mathcal{A}(I)_-$ and $A_{2-} \in \mathcal{A}(J)_-$ such that

$$\omega(A_{1-}A_{2-}) \neq 0.$$

Then there exists some product state $\omega_{1,i} \circ \omega_{2,i}$ in the decomposition such that

$$\omega_{1,i} \circ \omega_{2,i}(A_{1-}A_{2-}) \neq 0.$$

But this is impossible. Our assertion is now proved. □

For a given symmetry G, there may exist G-invariant separable states which have no separable decomposition that consists of all G-invariant product states [16], for example, $U(1)$-symmetry. The next proposition shows the nonexistence of such separable states for Θ-symmetry.

Proposition 2. *Let I and J be a pair of disjoint subsets and ω be an even state on $\mathcal{A}(I \cup J)$. If ω is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then it has a separable decomposition*

$$\omega = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}, \tag{11}$$

such that $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and all the marginal states $\omega_{1,i}$ on $\mathcal{A}(I)$ and $\omega_{2,i}$ on $\mathcal{A}(J)$ are even.

If I and J are finite subsets, all $\omega_{1,i}$ and $\omega_{2,i}$ above can be taken from the set of pure even states.

Proof. Let $\omega = \sum_i \lambda_i \omega_i$ where $\omega_i := \omega_{1,i} \circ \omega_{2,i}$, $\omega_{1,i}$ and $\omega_{2,i}$ are some states on $\mathcal{A}(I)$ and $\mathcal{A}(J)$. We shall show that all $\omega_{1,i}$ and $\omega_{2,i}$ can be taken from even states.

By proposition 1 at least one of $\omega_{1,i}$ and $\omega_{2,i}$ should be even for the existence of the product state $\omega_{1,i} \circ \omega_{2,i}$. For a given state ψ, let $\widehat{\psi}$ denote its Θ-averaged state $\frac{\psi + \psi\Theta}{2}$. By the evenness of ω, we have the following identity:

$$\omega = \widehat{\omega} = \sum_i \lambda_i \widehat{\omega}_i.$$

For each i, $\widehat{\omega}_i$ is an even product state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$ because $\widehat{\omega}_i = \widehat{\omega}_{1,i} \circ \widehat{\omega}_{2,i}$. Replacing $\omega_{1,i}$ and $\omega_{2,i}$ by $\widehat{\omega}_{1,i}$ and $\widehat{\omega}_{2,i}$, we obtain a separable decomposition for ω consisting of all even states.

For a finite-dimensional CAR system, every even state can be decomposed into an affine sum of pure even states. Hence, if I is finite, we have $\omega_{1,i} = \sum_{i(j)} l_{i(j)} \omega_{1,i(j)}$, where $l_{i(j)} > 0$, $\sum_{i(j)} l_{i(j)} = 1$, and each $\omega_{1,i(j)}$ is a pure even state of $\mathcal{A}(I)$. Similarly, $\omega_{2,i} = \sum_k l_{i(k)} \omega_{2,i(k)}$, where $l_{i(k)} > 0$, $\sum_{i(k)} l_{i(k)} = 1$, and each $\omega_{2,i(k)}$ is a pure even state of $\mathcal{A}(J)$. Hence, we have an even-pure-state decomposition $\omega_{1,i} \circ \omega_{2,i} = \sum_{i(j),i(k)} l_{i(j)} l_{i(k)} \omega_{1,i(j)} \circ \omega_{2,i(k)}$ for each i. Those for all indices induce a desired decomposition of ω. □

For the second statement of this proposition, the assumption that I and J are finite subsets is necessary since there is an even state that is pure on $\mathcal{A}(I)_+$ but non-pure on $\mathcal{A}(I)$ when $|I|$ is infinite [17].

Remark 1. Examples of bosonic $U(1)$-gauge invariant separable states that cannot be prepared locally under the $U(1)$-gauge symmetry are given in [16]. We now consider the lattice-fermionic counterpart of example 1 (equation (4)) given there. Let $|0\rangle$ and $|1\rangle$ be the unit vector denoting the absence and the presence of one-fermion particle. Let two disjoint subsystems under consideration be indicated by A and B. Let

$$\rho_1 := \tfrac{1}{4}(|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + |1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1|) + 1/2|\Psi_+\rangle_{AB}\langle\Psi_+|, \tag{12}$$

where $|\Psi_+\rangle_{AB} := \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$. Let $|a_{1,2}\rangle = |b_{1,2}\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|a_{3,4}\rangle = |b_{3,4}\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm \mathrm{i}|1\rangle)$, where a and b indicate that the states are of A and B, respectively, and the subscripts 1 and 2 correspond to + and −, respectively.

For the bosonic case, ρ_1 is separable since it has its separable decomposition: $\rho_1 = \sum_{k=1}^{4} |a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|$.

For the fermionic case, ρ_1 is non-separable. Note that the notation $|a_k\rangle\langle a_k| \otimes |b_k\rangle\langle b_k|$ ($k = 1, 2, 3, 4$) makes no sense (even mathematically), because there is no product-state extension for $|a_k\rangle\langle a_k|$ on A and $|b_k\rangle\langle b_k|$ on B that are both noneven states. (In fact, there is no state extension at all for them by theorem 1 (2) of [13].) Furthermore, proposition 1 claims the nonexistence of separable decomposition of ρ_1 due to the particle hopping correlation by $|\Psi_+\rangle_{AB}\langle\Psi_+|$. As noted in [18], non-separability due to (purely) fermion hopping cannot be distilled to use because of the restriction of local operations by the univalence superselection rule.

Let $\mathcal{A}(I)'$ ($\mathcal{A}(J)'$) denote the commutant algebra of $\mathcal{A}(I)$ ($\mathcal{A}(J)$) in $\mathcal{A}(I \cup J)$. If the cardinality $|I|$ of I is infinite, $\mathcal{A}(I)' = \mathcal{A}(J)_+$. If $|I|$ is finite, $\mathcal{A}(I)' = \mathcal{A}(J)_+ + v_I\mathcal{A}(J)_-$ and $\mathcal{A}(I \cup J) = \mathcal{A}(I) \otimes \mathcal{A}(I)'$ hold. As is well known, the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$ is transformed to the tensor-product pair $(\mathcal{A}(I), \mathcal{A}(I)')$ and to $(\mathcal{A}(J), \mathcal{A}(J)')$ by Jordan–Klein–Wigner transformations. We consider how the properties of state correlation (separability, entanglement degrees, etc) will remain or change by the replacement of the CAR pair by the tensor product ones and vice versa. The following proposition shows that the separability condition for the CAR pair always implies that for the tensor-product pair for even states. We have noted in remark 1 that the converse of this proposition does not hold. Later in proposition 7 we will see that the evenness assumption is unnecessary. We now provide a simple proof that makes use of the evenness assumption.

Proposition 3. *Let I and J be a pair of disjoint subsets and ω be an even state on $\mathcal{A}(I \cup J)$. If it is separable for the CAR pair $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then so it is for the tensor-product pair $\mathcal{A}(I)$ and $\mathcal{A}(I)'$.*

Proof. Since ω is an even separable state, it has a separable decomposition in the form of (11) where each $\omega_{1,i}$ and $\omega_{2,i}$ is even. By CARs and the evenness of $\omega_{1,i}$ and $\omega_{2,i}$, we verify that $\omega_{1,i} \circ \omega_{2,i}$ is a product state with respect to the tensor-product pair $\mathcal{A}(I)$ and $\mathcal{A}(I)'$. Hence, the separability of ω for the pair $(\mathcal{A}(I), \mathcal{A}(I)')$ follows. □

3. The entanglement formation under the univalence superselection rule

We introduce a quantity that measures non-separability of even states between $\mathcal{A}(I)$ and $\mathcal{A}(J)$ for disjoint finite subsets I and J. The von Neumann entropy of the density matrix D is given by

$$-\mathbf{Tr}(D \log D), \tag{13}$$

where **Tr** denotes the trace which takes the value 1 on each minimal projection. The von Neumann entropy of a state ω is given by (13) for its density matrix with respect to **Tr** and is denoted by $S(\omega)$.

For even state ω of $\mathcal{A}(I \cup J)$, we define

$$E^{\Theta}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) := \inf_{\omega=\sum \lambda_i \omega_i^{\mathrm{e}}} \sum_i \lambda_i S\big(\omega_i^{\mathrm{e}}\big|_{\mathcal{A}(I)}\big), \tag{14}$$

where the infimum is taken over all even-state decompositions of ω. Namely, each ω_i^{e} is an even state on $\mathcal{A}(I \cup J)$. We shall call this quantity *entanglement of formation under the univalence superselection rule*. From [19], it follows that

$$S\big(\omega_i^{\mathrm{e}}\big|_{\mathcal{A}(I)}\big) = S\big(\omega_i^{\mathrm{e}}\big|_{\mathcal{A}(J)}\big) = S\big(\omega_i^{\mathrm{e}}\big|_{\mathcal{A}(I)_+}\big) = S\big(\omega_i^{\mathrm{e}}\big|_{\mathcal{A}(J)_+}\big). \tag{15}$$

Thus, the subsystem in the rhs of (14) can be any of $\mathcal{A}(I)$, $\mathcal{A}(J)$, $\mathcal{A}(I)_+$ and $\mathcal{A}(J)_+$. We give a criterion of the separability between the CAR pair $\mathcal{A}(I)$ and $\mathcal{A}(J)$ in terms of this degree.

Proposition 4. *Let I and J be finite disjoint subsets and ω be an even state of $\mathcal{A}(I \cup J)$. It is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$ if and only if its entanglement formation under the univalence superselection rule $E^{\Theta}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ is equal to zero.*

Proof. If an even state ω satisfies the separability condition, then by proposition 2 there exists a product-state decomposition

$$\omega(A_1A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1A_2) \tag{16}$$

such that each of $\omega_{1,i}$ and $\omega_{2,i}$ is even and pure. Thus, $E^{\Theta}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = 0$ by definition. The converse direction is easily verified. □

4. Non-separable for the CAR pair but separable for the tensor-product pair

We construct some $U(1)$-gauge invariant states that are separable for the tensor-product pair $(\mathcal{A}(I), \mathcal{A}(I)')$ but non-separable for the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$. As will be specified below, their non-separability is purely due to fermion hopping terms.

Let τ be the tracial state on $\mathcal{A}(I \cup J)$. We note the following product properties of the tracial state:

$$\tau(A_1A_2) = \tau(A_1)\tau(A_2), \tag{17}$$

for every $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$, and

$$\tau(A_1B_2) = \tau(A_1)\tau(B_2), \qquad \tau(B_1A_2) = \tau(B_1)\tau(A_2), \tag{18}$$

for every $A_1 \in \mathcal{A}(I)$, $B_2 \in \mathcal{A}(I)'$ and every $B_1 \in \mathcal{A}(J)'$, $A_2 \in \mathcal{A}(J)$.

Let K_1 and K_2 be odd elements in $\mathcal{A}(I)_-$ and in $\mathcal{A}(J)_-$. Typically, those are field operators on specified regions. Let $K := 1/2\big(K_1^{\dagger}K_2 - K_1K_2^{\dagger}\big)$, which is self-adjoint and may represent fermion hopping. Suppose that $\|K_1\| \leqslant 1\|K_2\| \leqslant 1$, then $\|K\| \leqslant 1$. For $\lambda \in \mathbb{R}$, define

$$P(\lambda) := \mathrm{id} + \lambda K. \tag{19}$$

By definition $P(\lambda)$ is self-adjoint, and by

$$\|\lambda K\| \leqslant |\lambda|,$$

it is a positive operator if $|\lambda| \leqslant 1$. From (17) and the evenness of the tracial state, it follows that

$$\begin{aligned}\tau(P(\lambda)) &= \tau(\mathrm{id} + \lambda K) = \tau\left(\mathrm{id} + \frac{\lambda}{2}\big(K_1^\dagger K_2 - K_1 K_2^\dagger\big)\right) \\ &= \tau(\mathrm{id}) + \frac{\lambda}{2}\big(\tau\big(K_1^\dagger K_2\big) - \tau\big(K_1 K_2^\dagger\big)\big) \\ &= \tau(\mathrm{id}) + \frac{\lambda}{2}\big(\tau\big(K_1^\dagger\big)\tau(K_2) - \tau(K_1)\tau\big(K_2^\dagger\big)\big) \\ &= \tau(\mathrm{id}) + \frac{\lambda}{2}\cdot 0 = 1. \end{aligned} \tag{20}$$

Hence, for $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$, $P(\lambda)$ is a density matrix with respect to the tracial state τ. Let us define the state φ_λ on $\mathcal{A}(I \cup J)$ by

$$\varphi_\lambda(A) := \tau(P(\lambda)A), \qquad A \in \mathcal{A}(I \cup J). \tag{21}$$

By definition,

$$\Theta(P(\lambda)) = P(\lambda),$$

hence φ_λ is an even state of $\mathcal{A}(I \cup J)$.

We now compute the expectation value of φ_λ for the product element $A_1 A_2$ of $A_1 \in \mathcal{A}(I)$ and $A_2 \in \mathcal{A}(J)$. We have

$$\begin{aligned}\tau\big(\big(K_1^\dagger K_2\big)A_1A_2\big) &= \tau\big(K_1^\dagger(K_2A_1)A_2\big) \\ &= \tau\big(K_1^\dagger(\Theta(A_1)K_2)A_2\big) = \tau\big(\big(K_1^\dagger\Theta(A_1)\big)(K_2A_2)\big) \\ &= \tau\big(\big(K_1^\dagger\Theta(A_1)\big)\tau(K_2A_2) \\ &= \tau\circ\Theta\big(\Theta\big(K_1^\dagger\big)A_1\big)\tau(K_2A_2) \\ &= \tau\big(\Theta\big(K_1^\dagger\big)A_1\big)\tau(K_2A_2) \\ &= \tau\big(\big(-K_1^\dagger\big)A_1\big)\tau(K_2A_2) \\ &= -\tau\big(K_1^\dagger A_1\big)\tau(K_2A_2),\end{aligned}$$

and similarly

$$\tau\big(\big(K_1K_2^\dagger\big)A_1A_2\big) = -\tau(K_1A_1)\tau\big(K_2^\dagger A_2\big),$$

where we have used CARs, (5), (17) and $\tau = \tau\circ\Theta$ which follows from the uniqueness of the tracial state. Thus, we obtain

$$\varphi_\lambda(A_1A_2) = \tau(A_1A_2) - \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_1\big)\tau(K_2A_2) - \tau(K_1A_1)\tau\big(K_2^\dagger A_2\big)\big).$$

Since the tracial state is an even product state and $K_1 \in \mathcal{A}(I)_-$, $K_2 \in \mathcal{A}(J)_-$, writing $A_1 = A_{1+} + A_{1-}$, $A_{1\pm} \in \mathcal{A}(I)_\pm$, $A_2 = A_{2+} + A_{2-}$, $A_{2\pm} \in \mathcal{A}(J)_\pm$, we obtain

$$\varphi_\lambda(A_1A_2) = \tau(A_{1+})(A_{2+}) - \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_{1-}\big)\tau(K_2A_{2-}) - \tau(K_1A_{1-})\tau\big(K_2^\dagger A_{2-}\big)\big).$$

Similarly, we have

$$\begin{aligned}\varphi_\lambda(A_2A_1) &= \tau(A_1A_2) + \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_1\big)\tau(K_2A_2) - \tau(K_1A_1)\tau\big(K_2^\dagger A_2\big)\big) \\ &= \tau(A_{1+})(A_{2+}) + \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_{1-}\big)\tau(K_2A_{2-}) - \tau(K_1A_{1-})\tau\big(K_2^\dagger A_{2-}\big)\big).\end{aligned}$$

We summarize the above computations as follows.

Proposition 5. *The state φ_λ, given by the density $P(\lambda) := \mathrm{id} + \mathrm{i}\lambda K$ with $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$, $K := 1/2\big(K_1^\dagger K_2 - K_1 K_2^\dagger\big)$, $K_1 \in \mathcal{A}(I)_-$ and $K_2 \in \mathcal{A}(J)_-$ such that $\|K_1\| \leqslant 1$ and $\|K_2\| \leqslant 1$, has the following correlation functions:*

$$
\begin{aligned}
\varphi_\lambda(A_1A_2) &= \tau(A_{1+})(A_{2+}) - \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_{1-}\big)\tau(K_2A_{2-}) - \tau(K_1A_{1-})\tau\big(K_2^\dagger A_{2-}\big)\big),\\
\varphi_\lambda(A_2A_1) &= \tau(A_{1+})(A_{2+}) + \frac{\lambda}{2}\big(\tau\big(K_1^\dagger A_{1-}\big)\tau(K_2A_{2-}) - \tau(K_1A_{1-})\tau\big(K_2^\dagger A_{2-}\big)\big).
\end{aligned}
\tag{22}
$$

Let us recall a well-known criterion of separability for tensor-product systems in [20]: a state is separable for a bipartite tensor-product system $\mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if it is mapped to a positive element under $\Lambda \otimes \mathrm{id}$ for any positive map Λ from $\mathcal{A}_1$ to $\mathcal{A}_2$. By applying this criterion to the density (19) of φ_λ, we verify that it is separable for $(\mathcal{A}(I), \mathcal{A}(I)')$ and also for $(\mathcal{A}(J), \mathcal{A}(J)')$ for any $\lambda \in \mathbb{R}$, $|\lambda| \leqslant 1$. But this is not the case for the CAR pair $(\mathcal{A}(I), \mathcal{A}(J))$. Take one-site subsets $I = \{1\}$ and $J = \{2\}$, and let $K_1 = a_1$, $K_2 = a_2$ for computational simplicity. Then, we have

$$
\varphi_\lambda(a_1^*a_2) = \frac{\lambda}{8} \qquad \varphi_\lambda(a_1a_2^*) = -\frac{\lambda}{8}. \tag{23}
$$

By proposition 1, φ_λ is non-separable between $\mathcal{A}(I)$ and $\mathcal{A}(J)$ for any non-zero λ.

5. The general case including noneven states

In this section, our state ω on $\mathcal{A}(I \cup J)$ can be noneven. We define the following quantity for positive number k, $0 \leqslant k \leqslant 1$:

$$
E^k_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) := \inf_{\omega=\sum \lambda_i\omega_i} \sum_i \lambda_i(kS(\omega_i|_{\mathcal{A}(I)}) + (1-k)(\omega_i|_{\mathcal{A}(J)})), \tag{24}
$$

where the infimum is taken over all the state decompositions of ω in the state space of $\mathcal{A}(I \cup J)$. For any pure state ω of $\mathcal{A}(I \cup J)$, it reduces to

$$
E^k_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = kS(\omega|_{\mathcal{A}(I)}) + (1-k)S(\omega|_{\mathcal{A}(J)}). \tag{25}
$$

For $k = 1, 0$, (24) reduces to the usual definition of entanglement formation [21] denoted as $E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I))$ and $E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(J))$, respectively. We note that $E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I))$ quantifies the non-separability of states for the tensor-product pair $(\mathcal{A}(I), \mathcal{A}(I)')$, not for our target $(\mathcal{A}(I), \mathcal{A}(J))$.

Asymmetry of entanglement may arise for noneven states as shown in [22]. For example, there is a noneven pure state ϱ on $\mathcal{A}(I \cup J)$ such that $\varrho|_{\mathcal{A}(I)}$ is a pure state while $\varrho|_{\mathcal{A}(J)}$ is a tracial state, giving

$$
0 = S(\varrho|_{\mathcal{A}(I)}) < S(\varrho|_{\mathcal{A}(J)}) = \log 2 \tag{26}
$$

when $I = \{1\}$ and $J = \{2\}$. Hence, for quantification of state correlation between $(\mathcal{A}(I), \mathcal{A}(J))$ for noneven states, we have to take both subsystems on I and on J into account. Here, we take the equal probability $k = 1/2$ for simplicity and denote $E^{1/2}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ by $E^{\mathrm{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ which is called the averaged entanglement formation.

Proposition 6. *Let I and J be finite disjoint subsets and ω be a state on $\mathcal{A}(I \cup J)$. Then it is a separable state for $\mathcal{A}(I)$ and $\mathcal{A}(J)$ if and only if the averaged entanglement formation $E^{\mathrm{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ is equal to zero.*

Proof. If ω satisfies the separability condition (8), then there exists the product-state decomposition

$$\omega(A_1 A_2) = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i}(A_1 A_2). \tag{27}$$

For each index i, at least one of $\omega_{1,i}$ and $\omega_{2,i}$ should be even for the existence of the product state $\omega_{1,i} \circ \omega_{2,i}$ by proposition 1. So let $\omega_{1,i}$ be even. Then, it can be decomposed as $\omega_{1,i} = \sum_j l_{i(j)} \omega_{1,i(j)}$, where $l_{i(j)} > 0$, $\sum_j l_{i(j)} = 1$, and all $\omega_{1,i(j)}$ can be taken from pure even states of $\mathcal{A}(I)$. (This is always possible when I is finite.) We have a decomposition of $\omega_{2,i}$ as $\omega_{2,i} = \sum_k l_{i(k)} \omega_{2,i(k)}$, where $l_{i(k)} > 0$, $\sum_k l_{i(k)} = 1$, and all $\omega_{2,i(k)}$ are pure states of $\mathcal{A}(J)$. Since each $\omega_{1,i(j)}$ is an even state of $\mathcal{A}(I)$, we are given the (unique) product-state extension $\omega_{1,i(j)} \circ \omega_{2,i(k)}$ for any $i(j)$ and $i(k)$. Repeating the same machinery for all i, we have a state decomposition of ω into $\{\omega_{1,i(j)} \circ \omega_{2,i(k)}\}$ where each $\omega_{1,i(j)}$ and $\omega_{2,i(k)}$ is a pure state. Hence,

$$S(\omega_{1,i(j)}|_{\mathcal{A}(I)}) = S(\omega_{2,i(k)}|_{\mathcal{A}(J)}) = 0$$

for every $i(j), i(k)$. Thus, this decomposition gives

$$E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = 0. \tag{28}$$

Conversely, assume (28). By definition, there exists a state decomposition $\omega = \sum_i \lambda_i \omega_i$ such that

$$S(\omega_i|_{\mathcal{A}(I)}) = S(\omega_i|_{\mathcal{A}(J)}) = 0, \tag{29}$$

for all i. This implies that ω_i has pure state restrictions on both $\mathcal{A}(I)$ and $\mathcal{A}(J)$. By theorem 1 (2) in [13], at least one of $\omega_i|_{\mathcal{A}(I)}$ and $\omega_i|_{\mathcal{A}(J)}$ should be even for the existence of their state extension ω_i on $\mathcal{A}(I \cup J)$ and ω_i is uniquely given as $\omega_i|_{\mathcal{A}(I)} \circ \omega_i|_{\mathcal{A}(J)}$. Hence, ω can be written as the affine sum of the product states $\{\omega_i\}$ and hence is a separable state. □

The following relationships among the introduced degrees are obvious.

Lemma 7. *For any state ω on $\mathcal{A}(I \cup J)$,*

$$E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant 1/2 E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I)) + 1/2 E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(J)). \tag{30}$$

For any even state ω on $\mathcal{A}(I \cup J)$,

$$E^{\ominus}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)), \tag{31}$$

and

$$E^{\ominus}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I)), E^{\ominus}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(J)). \tag{32}$$

Proof. The inequality (30) follows directly from the definitions. The optimal decomposition of $E^{\ominus}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ is given by some $\omega = \sum \lambda_i \omega_i^{\mathrm{e}}$ such that all ω_i^{e} are pure and even. Since each ω_i^{e} satisfies (15), (31) and (32) follow. □

The inequalities (30) and (32) are exact for φ_λ of $\lambda \neq 1$ in section 4, since it is always separable for $(\mathcal{A}(I), \mathcal{A}(I)')$ and for $(\mathcal{A}(J), \mathcal{A}(J)')$ hence $E_{\mathcal{A}(I\cup J)}(\varphi_\lambda, \mathcal{A}(J)) = E_{\mathcal{A}(I\cup J)}(\varphi_\lambda, \mathcal{A}(I)) = 0$, while for the case of (23) it is non-separable for $(\mathcal{A}(I), \mathcal{A}(J))$ and hence both $E^{\ominus}_{\mathcal{A}(I\cup J)}(\varphi_\lambda, \mathcal{A}(I), \mathcal{A}(J))$ and $E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\varphi_\lambda, \mathcal{A}(I), \mathcal{A}(J))$ should be non-zero.

The noneven pure state ϱ with its asymmetric marginal states (26) gives $E_{\mathcal{A}(I\cup J)}(\varrho, \mathcal{A}(J)) = 0$, $E_{\mathcal{A}(I\cup J)}(\varrho, \mathcal{A}(I)) = \log 2$ and $E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\varrho, \mathcal{A}(I), \mathcal{A}(J)) = 1/2(\log 2)$. Hence,

$$E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I))$$

and

$$E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) \geqslant E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(J))$$

are not satisfied in general.

We can now generalize proposition 3 to the case including noneven states, assuming additionally that the systems are finite dimensional.

Proposition 7. *Let I and J be finite subsets and ω be a state on $\mathcal{A}(I \cup J)$. If it is separable for the CAR pair $\mathcal{A}(I)$ and $\mathcal{A}(J)$, then so it is for the tensor-product pair $\mathcal{A}(I)$ and $\mathcal{A}(I)'$.*

Proof. If it is separable, then $E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J)) = 0$. Hence by (30), $E_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I)) = 0$. This is equivalent to the separability of ω for $(\mathcal{A}(I), \mathcal{A}(I)')$. □

By propositions 4 and 6, both $E^{\Theta}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ and $E^{\text{avr.}}_{\mathcal{A}(I\cup J)}(\omega, \mathcal{A}(I), \mathcal{A}(J))$ serve characterization of separable states for $(\mathcal{A}(I), \mathcal{A}(J))$. We do not know whether the inequality (31) can be strict.

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